Problem Theory applied to Model Theory and Logic

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September 7, 2024

Abstract

Concepts from Agama's papers on Problem Theory are used to characterize and develop some basic theorems of Model Theory and Logic. We "represent" core concepts in Logic with problems and solutions. These include Godel's First Incompleteness Theorem, Godel's Completeness Theorem, and the Compactness Theorem. Further developments in this topic will be pursued.

1 Introduction

Theophilus Agama authored 2 papers, one called "On the theory of problems and their solution spaces" [1], and the other, "On the topology of problems and their solutions" [2]. Readers are advised to read [1] first. As both papers deal with the same concept(s), problem and solution spaces, I have decided to call those papers as belonging to "Problem Theory". For now we are not interested in the topology of Problem Theory, but are interested in characterizing and developing some small problems in Model Theory in terms of Problem Theory. As of August 2024, besides this paper and [1] and [2], there have been no other papers published on Problem Theory. Also, [1] does have some "holes", in which I may address in this paper.

Lastly, note that problems and/or solutions are not defined; they are treated as fundamental objects akin to sets or axioms.

2 Core Definitions

All definitions here will be given in Problem Theory unless explicitly stated otherwise.

- Definition. A **solution to a problem** is a relation $\mathbf{S}$ such that $a\ \mathbf{S}\ b$ iff a is a solution and b is a problem, with a solving b. Such a relation need not be an equivalence relation.

- Definition. M is a **model** of σ iff for each solution in M, there exists a corresponding problem in σ, such that each solution in M solves a corresponding problem in σ.
- Definition. ϕ is **valid** iff for every problem in ϕ, ϕ has a solution in any model of a language L.
- Definition. A language is a set of problems and solutions.
- Definition. A **proof** of a problem P in L is sequence of solutions $X := (x_1, x_2, x_3, ...\ x_n)$ in L such that (a) $\forall x \in X$, $x \in S_{x_n}(P)$, (b) x_n is a solution to L, and (c) under a rule(s) of inference r, (x_1 r x_2 r x_3, ... x_{n-1}) is equivalent to x_n, and $x_{n-2}\ r\ x_{n-1}$ is true. P is said to be a **theorem** of L.
- Definition. A negative problem $\neg P$ is a problem such that it cannot be solved. A negative solution is a statement that is not a solution to any problem.

consisting of all solutions. Endow them with composition.

Note: the notion for the "actual" negation of problems and solutions cannot be formed, as there does not exist a formal definition for problems and solutions, but Agama's work is not limited.

3 Some elementary theorems in Model Theory and Logic

We start by proving a practice problem in Chang and Keisler (p.16) [3], to demonstrate our new system. A sentence is satisfiable if it has at least one model.

Theorem 1. A sentence ϕ is satisfiable iff $\neg\phi$ is not valid.

Note that as the negation of a problem or solution may not be defined, we use "negative" problems and solutions.

Proof. ($\leftarrow$) If ϕ is a problem, or set of problems, then if "ϕ cannot be solved" is invalid, then ϕ can be solved, proving this case. ($\rightarrow$) If $\neg\phi$ is not valid, then it may be solved, therefore $\neg\phi$ has a model. □

Another, more important theorem is proved.

Theorem 2. Lindenbaum's Theorem.

Proof. Let Σ represent a set of problems, and that there is no 2 problems σ_1 and σ_2 with solutions τ_1 and τ_2 in Σ such that $S_{\sigma_1}(\tau_1)$ and $S_{\sigma_2}(\tau_2)$ do not have "contradictory" solutions. The rest continues in a way similar to Chang and Keisler (p. 10). □

An alternate proof, and equivalence of Godel's First Incompleteness Theorem is given.

Theorem 3. "There exists a problem with no solution" is equivalent to Godel's First Incompleteness Theorem.

Proof. ($\rightarrow$) Let there exist a problem P in a language L, in which arithmetic can be carried out in, with no solution. As a problem P may have a solution iff it has a proof, P has no proof in L, implying Godel's First Incompleteness Theorem. ($\leftarrow$) Let there exist a statement S in a language L with no proof, represent S as a problem, as a problem may have a solution iff it has a proof, S has no solution. □

One may also prove Godel's completeness theorem (as well as its extended form) using the definitions and methods we have developed in this paper.

Proof of the Compactness Theorem (countable case).

Proof. Let every problem of a subset of a set of problems $\mathcal{P}$ have a model. Then every problem of a subset of a set of problems in $\mathcal{P}$ has its corresponding set of solutions. Taking the union of the corresponding sets of solutions and subset of sets of problems in $\mathcal{P}$, we have arrived at the compactness theorem.

□

4 Motivation of Problem Theory

The subject in which I call "Problem Theory" was developed by T. Agama in 2 papers, in which in his first, he states the original motivation of Problem Theory was to address and characterize P = NP. However, I even with its original motivation, I purpose that it could be used to develop, in addition to Model Theory, Proof Theory, among other branches of Logic. Additionally, Problem Theory allowed us to prove existing problems in Logic more efficiently.

5 Holes in Agama's foundational paper

Some missteps in Agama's foundational paper [1] are listed and addressed.

- In the proof of Theorem 2.6 in [1] ("If $\mathcal{P}(X)$ is the induced problem space of providing solution X to problem Y, then $Y \in \mathcal{P}_Y(X)$."), it is not elaborated on how an infinite decreasing sequences of the entropy of solution spaces tending towards zero is not possible.

- In the proof of Theorem 3.2 in [1] ("There exists a problem with no solution"), it is again not elaborated on how an infinite chain of sub-covers of smaller problem spaces are contradictory.

An alternate proof of Theorem 3.2 in [1] involves a "diagonalization argument" akin to Cantor's proof of the uncountability of real numbers. Theorem 2.6 in [1] is an immediate consequence of the definition of problem spaces.

References

[1] Agama, Theophilus. (2022) "On the theory of problems and their solution spaces"
[2] Agama, Theophilus. (2023) "On the topology of problems and their solutions"
[3] Chang, C.C, and Keisler, H. Jerome. *Model Theory.* 3rd ed, Dover, 2020.